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**A note on the solution of
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complex coefficients related to
aerial null-filling calculations**

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Research Report No. RA-7 (1967/36)

A NOTE ON THE SOLUTION OF ALGEBRAIC EQUATIONS WITH COMPLEX COEFFICIENTS RELATED TO AERIAL NULL-FILLING CALCULATIONS

Coaxal Circles and Smith Charts

The references to Smith charts on page 2 of this report are misleading. The values of z which satisfy Equation (11) lie on a circle whose centre is where $z = (1 + \rho^2)/(1 - \rho^2)$ and radius $2\rho/|1 - \rho^2|$. If $\rho = 0$ this gives $z = 1$; as ρ increases from 0 to 1, the circle, whose centre moves to the right along the real axis, increases in radius but remains to the right of the imaginary axis; the circle degenerates into the imaginary axis as $\rho \rightarrow 1$ and for greater values of ρ it lies to the left of the imaginary axis, and dwindles as ρ increases until as $\rho \rightarrow \infty$ it reduces to the point $z = -1$. The length of the tangent from any point $z = jk$ on the imaginary axis to any of these circles is $(k^2 + 1)^{1/2}$ whatever the value of ρ may be, and hence they are correctly described as coaxal; the radical axis is the imaginary axis. Further, any circle through the points $z = \pm 1$ cuts any of the circles given by Equation (11) for any value of ρ at right angles. The Smith chart circles are a limiting case of the above-mentioned systems when the scale is reduced without limit, so that the circles satisfying Equation (11) are reduced to circles touching the imaginary axis at the origin, and the circles through the points $z = \pm 1$ become circles touching the real axis at the origin.

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J.W. Head, M.A., F.Inst.P., C.Eng., M.I.E.E., F.I.M.A.



for Head of Research Department

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**A NOTE ON THE SOLUTION OF ALGEBRAIC EQUATIONS WITH COMPLEX
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A NOTE ON THE SOLUTION OF ALGEBRAIC EQUATIONS WITH COMPLEX COEFFICIENTS RELATED TO AERIAL NULL-FILLING CALCULATIONS

SUMMARY

One method of broadcast transmitter aerial null-filling consists essentially in solving an algebraic equation (which in general is of degree 20 or more and has complex coefficients) and inverting some of the roots with respect to the unit circle in the complex plane. Such inversion does not alter the radiation pattern, but may make the practical realization of the aerial easier. Hill¹ has also explored possibilities of improving undesirable radiation-pattern characteristics by appropriate adjustment of these roots. The co-phased quarter-sine amplitude distribution is first discussed in detail; this case is specially simple because the above-mentioned algebraic equation has real coefficients and a straightforward graphical technique of solution is available. Another important practical case is when the roots of the algebraic equation are nearly equally spaced round the unit circle in the complex plane. The general case is outlined; if the given equation can be regarded as a variant of one whose roots are already known (or more easily found) the procedure is simplified.

1. INTRODUCTION

Various possible methods of aerial null-filling are available, and several of them are fully discussed by Hill¹. In one of these methods, the equation

$$f_n(z) = I_1 z^{n-1} + I_2 z^{n-2} + I_3 z^{n-3} + \dots + I_n = 0 \quad (1)$$

plays a very important part, where there are n elements altogether, and I_r is the current in the r th of these. In general, the elements will not be co-phased, and therefore the coefficients I_r must be regarded as complex, while n may be in the region 20 to 30. Thus it appears as if we may be required to solve an equation of high degree having complex roots. In this note, we first consider the co-phased quarter-sine amplitude distribution, which provides extremely smooth null-filling but has the disadvantage that the ratio of maximum to minimum current in the tiers is inconveniently large. For this case

$$I_r = \sin \{(2r-1)\pi/4n\} \quad (2)$$

so that all the I_r are real. This means that the equation (1) can be solved by a library routine on the Research Department Elliott 803B computer for an adequate value of n , and in this case a graphical method of solution is also available, as fully discussed for the particular case $n = 8$ in Section 2. Another case of considerable practical interest is

that in which the zero-order approximation to the required $f_n(z)$ has

$$I_1 = I_n = 1, I_2 = I_3 = \dots = I_{n-1} = 0 \quad (3)$$

and the required $f_n(z)$ is built up by using Hill's technique¹ of "perturbing" some of the roots of (1) radially; this case is discussed in Section 3. Section 4 outlines ways in which the general case of solving (1) for complex I_r and high n could be tackled, with particular attention to the fact that it is much easier to adjust the roots of (1) to achieve some desirable end and deduce the coefficients than vice versa. In particular, it must be noted that for the null-filling application, replacing any root of (1) by its inverse with respect to the unit circle in the complex z -plane does not alter the radiation pattern of the array, but it will of course alter $f_n(z)$ and it may significantly improve the possibility of realizing the radiation pattern to replace several roots of $f_n(z) = 0$ by their inverses. If all the roots of $f_n(z) = 0$ are replaced by their inverses, we replace I_r by I_{n+1-r} and thus do not effectively alter the current distribution, but even allowing for this, there are 2^{n-1} effectively different possibilities of changes by inversion to be taken into consideration.

A routine for solving (1) in the general case when $n = 5$ has been written and fully tested on the Research Department computer, but unfortunately 5 is not a sufficiently high value of n for the null-filling application.

2. THE CO-PHASED QUARTER-SINE AMPLITUDE DISTRIBUTION

All detailed work will be done for the case in which $n = 8$, but the procedure is easily generalized for any other value of n . When $n = 8$, (1) becomes

$$z^7 \sin \frac{\pi}{32} + z^6 \sin \frac{3\pi}{32} + z^5 \sin \frac{5\pi}{32} + \dots + z \sin \frac{13\pi}{32} + \sin \frac{15\pi}{32} = 0 \quad (4)$$

Multiply through (4) by $2j$ and let

$$\xi = e^{j\pi/32} \quad (5)$$

We have

$$z^7 \xi \left(1 + \frac{\xi^2}{z} + \frac{\xi^4}{z^2} + \dots + \frac{\xi^{14}}{z^7} \right) = \frac{z^7}{\xi} \left(1 + \frac{1}{\xi^2 z} + \frac{1}{\xi^4 z^2} + \dots + \frac{1}{\xi^{14} z^7} \right) \quad (6)$$

Summing the geometric progressions on both sides of (6)

$$\xi^2 \frac{1 - (\xi^{16}/z^8)}{1 - (\xi^2/z)} = \frac{1 - (1/\xi^{16}z^8)}{1 - (1/\xi^2 z)} \quad (7)$$

and when (7) is cleared of fractions and the value of ξ from (5) is taken into account, we finally obtain

$$z^9 + z^8 = (z - 1) \cot \frac{\pi}{32} = 10.153(z - 1) \quad (8)$$

Now equation (8) is of degree 9 whereas equation (4) is of degree 7; the two extra roots are

$$z = \xi^2 = e^{j\pi/16}; \quad z = \xi^{-2} = e^{-j\pi/16} \quad (9)$$

The remaining roots of (8) are roots of (4) also; they were obtained by the Research Department computer (in about a minute) as

$$\left. \begin{aligned} &0.71674889 \pm 1.0025036j \\ &-0.21412418 \pm 1.3446045j \\ &-1.1814887 \pm 0.92333381j \\ &-1.6038395 \end{aligned} \right\} \quad (10)$$

The following method of attack would permit these roots to be found graphically, by elementary procedures. Let

$$\rho = \left| \frac{z - 1}{z + 1} \right| \quad (11)$$

so that the loci along which ρ is constant are the coaxial* circles drawn on a Smith chart. Then (8) and (11) can be combined to give

$$|z| = [10.153\rho]^{1/8} = \xi_0 \text{ say} \quad (12)$$

and if $\arg z = \theta$ it can also be shown that

$$\cos \theta = k \frac{1 - \rho^2}{1 + \rho^2} \text{ where } k = [\xi_0 + (1/\xi_0)]/2 \quad (13)$$

and, if $\phi = \arg(z - 1)$, $\psi = \arg(z + 1)$, $0 < \phi, \psi < \pi$

$$\tan \phi = \frac{\xi_0 \sin \theta}{\xi_0 \cos \theta - 1}; \quad \tan \psi = \frac{\xi_0 \sin \theta}{\xi_0 \cos \theta + 1} \quad (14)$$

The condition that z shall be a root of (1) is that

$$\arg\{z^8(z + 1)/(z - 1)\} = 8\theta - \phi + \psi = 2\mu\pi (\mu \text{ integer}) \quad (15)$$

as well as (12). $\mu = 0$ corresponds to the spurious roots (9), while the genuine roots (10), correspond to $\mu = 1, 2, 3, 4$ in that order.

Given ρ , ξ_0 and k are deducible immediately and thus θ , ϕ and ψ can be found from (13), (14) and (15). As $(8\theta - \phi + \psi)$ increases steadily with ρ , approximations to the roots (10) can easily be found by plotting $(8\theta - \phi + \psi)$ against ρ , and improved by interpolation. Alternatively, if plotting accuracy is sufficient, a value of ρ corresponding to a circle drawn on a Smith chart can be taken, and for this value of ρ the circle (12) can be drawn, with centre the origin and radius ξ_0 . At the intersection P of these two circles (in the upper half plane), θ , ϕ and ψ can all be measured by joining P to the origin and to $(\pm 1, 0)$, and hence θ , ϕ and ψ can be obtained and substituted into (15).

Hence for co-phased quarter-sine amplitude distributions the equation (1) takes a particularly simple form. This equation can be solved by means of a computer library routine for large values of n , but it can also be solved in an elementary and straightforward manner for any value of n by means of the transformation (10) and the graphical equivalent of using Smith charts.

3. ROOTS EVENLY DISTRIBUTED ON THE UNIT CIRCLE

Whatever the value of n , there are two cases in

* The locus of points from which equal tangents can be drawn to two circles is a straight line called the radical axis. A system of circles any two of which have the same radical axis is called coaxial.

which we can solve (1) at sight, namely

$$(i) I_1 = I_2 = I_3 = \dots = I_n$$

(ii) $I_1 = 1, I_n = R^n e^{j\phi}$ ($0 \leq \phi < 2\pi$; R real and positive) While $I_2, I_3, \dots, I_{n-1}$ are all zero.

In case (1) we have

$$f_n(z) = \frac{z^n - 1}{z - 1}; \quad z = e^{j2\mu\pi/n} \quad \mu = 1, 2, \dots, (n-1)$$

so that the roots are uniformly distributed round the unit circle, except in so far as the spurious root $z = 1$ is missing.

In case (ii) the roots are $z = R e^{j(2\nu\pi + \phi)/n}$ ($\nu = 1, 2, \dots, n$) which are uniformly distributed around a circle of radius R .

Unfortunately in both these cases there would be deep minima or even actual nulls in the radiation pattern; this is just what we wish to avoid. In case (i), inverting any root with respect to the unit circle leaves it unchanged, and the same is nearly true in case (ii) if $R \approx 1$ while if R is appreciably different from 1, the ratio of I_1 to I_n will become inconveniently large. Hill¹ has found that it is useful to alter appropriate roots radially, so that in case (ii) for example, the ν th root becomes $k_\nu R \exp\{j(2\nu\pi + \phi)/n\}$ where k_ν is a real constant, depending upon ν . Hill shows that only one point (say P_ν) of the radiation pattern is significantly affected if k_ν is altered while the other zeros of $f_n(z)$ are left unchanged; he shows how to calculate k_ν so that an appropriate change in the radiation pattern at P_ν will result. If therefore the k_ν are calculated for points P_ν where nulls are obnoxious and k_ν is left unity at places where they are not, we have a new set of roots of $f_n(z) = 0$ from which the corresponding coefficients of $f_n(z)$ can be deduced by mere multiplication - a tedious process if n is high but relatively simple compared with the solution of a high-degree equation with complex coefficients.

4. THE GENERAL CASE IN OUTLINE

In the general case, our best hope of obtaining a solution is to reduce it to the form

$$F_n(z) = G_n(z) \quad (16)$$

where $F_n(z) = 0$ is an equation whose roots we know (either because we have previously had to solve a similar but not identical equation, or because

$$F_n(z) = (z + \alpha)^n + \beta^n \quad (17)$$

so that the solution of $F_n(z) = 0$ is obvious) and $G_n(z)$ has coefficients which are as small as possible. One possibility in the absence of special information is to take the coefficients of $F_n(z)$ to be the real parts of the coefficients of the given equation, since the solution of algebraic equations with real coefficients by means of a digital computer is a standard subroutine which is usually satisfactory when the coefficients are not deliberately chosen so that multiple roots will be present. The roots of (16) can be obtained explicitly in terms of those of $F_n(z) = 0$ by an adaptation of Newton's method², and even when $|G_n(z)|$ is too large for this adaptation to be completely successful, we can regard the given equation as a variant of the equation whose roots are those thus found and repeat the process if necessary.

Iterative processes are likely to work well if the given equation can be reduced to the form (16) when $F_n(z)$ is given by (17) and n is moderate or large. For (16) can then be written for iteration in the form

$$z + \alpha = \beta e^{-(2\mu + 1)j\pi/n} \left[1 - \frac{G_n(z_K)}{\beta^n} \right]^{1/n} \quad (\mu \text{ integer}) \quad (18)$$

and we can take $z_0 = -\alpha$ (for all μ) if no better approximation is known. Aitken's well-known δ^2 -process is widely applicable to processes of this kind; if ξ_K, ξ_{K+1} and ξ_{K+2} are three successive approximations to $(z + \alpha)^n$ arising from the iterative process, then a better approximation still is

$$z' = \frac{\xi_K^2 + 1 - \xi_K \xi_{K+2}}{2\xi_{K+1} - \xi_K - \xi_{K+2}} \quad (19)$$

and the iterative process (18) can then be repeated with $z_0 + \alpha = (z')^{1/n}$ initially instead of 0. (19) can often give a useful result, leading to a true root of the given equation, even if the iterative process (18), considered in isolation, is divergent, and the value of μ is only relevant when $G_n(z_K)$ has to be determined. The biggest difficulty about the δ^2 -process (19) is that the numerator and denominator may both be very small, but as in (19) we are working with the n th power of the quantity really required, this difficulty is minimized.

5. CONCLUSIONS

Although, in the general case, solving an algebraic equation of high degree with complex coeffi-

cients is a formidable task, the equations associated with aerial null-filling tend to take one of several forms for which a relatively simple procedure is available. In particular, the co-phased quarter-sine amplitude distribution case is one which can be handled by elementary graphical procedures alone if necessary.

6. REFERENCES

1. BBC Research Department Report in preparation.
2. The effect of small variations in the coefficients of an equation upon its roots. BBC Research Department Report No. F-027, Serial No. 1966/57.

